

Collage:

Department:

Student Number:

Name:

- HW Set 4: Solve these problems at home (Due: 06/04).
- No calculators, books, and notes are permitted.
- Use the following formula if needed,

$$g_n(z) = \frac{1}{\Gamma(n)} \int_0^\infty \frac{x^{n-1}}{z^{-1}e^x - 1} dx = \sum_{k=1}^{\infty} \frac{z^k}{k^n}$$

$$\Gamma(n) = (n-1)!, \quad \frac{1}{2}! = \frac{\sqrt{\pi}}{2}$$

1. [20 pts.] Consider a system consists of n identical particles. For $n = 1$, it can be one of four states, x_1, x_2, x_3 , or x_4 . An identical particle system can be represented in two different ways, particle representation (PR) and occupation number representation (ONR). For example, in case of $n = 2$, ONRs $|1, 1, 0, 0\rangle_F$, $|1, 1, 0, 0\rangle_B$, and $|2, 0, 0, 0\rangle_B$ can be represented by PRs as

$$|1, 1, 0, 0\rangle_F = \frac{1}{\sqrt{2}} (|x_1, x_2\rangle - |x_2, x_1\rangle),$$

$$|1, 1, 0, 0\rangle_B = \frac{1}{\sqrt{2}} (|x_1, x_2\rangle + |x_2, x_1\rangle),$$

$$|2, 0, 0, 0\rangle_B = |x_1, x_1\rangle,$$

where $|n_1, n_2, n_3, n_4\rangle_{F[B]}$ is an ONR of a Fermion [Boson] system. Write PRs for $|1, 1, 0, 1\rangle_F$ and $|2, 0, 1, 0\rangle_B$

2. [30 pts.] For a 3D ideal (spinless) boson system with volume V , the average number of particles in excited states N_{ex} is given by

$$N_{ex} = \frac{V}{\lambda^3} g_{\frac{3}{2}}(z)$$

where $\lambda = h/\sqrt{2\pi mT}$.

- (a) Show that there is non-zero temperature T_c given by

$$T_c = \frac{h^2}{2\pi Bm} \left(\frac{N}{V} \right)^{2/3}$$

below which the number of particles in the ground state N_0 increases as N and find the constant B in terms of $\zeta\left(\frac{3}{2}\right) = g_{\frac{3}{2}}(1) \approx 2.6$.

- (b) Estimate the number of particles in a volume $V = 1 \text{ cm}^3$ for T_c to be $1 \mu\text{K}$ when the mass of a boson particle is $m = 23 \text{ GeV}/c^2$. Use $\hbar c = 1973 \text{ eV}\text{\AA}$ and $1 \text{ eV} = 11,000 \text{ K}$ if needed.

- (c) Discuss a way to obtain a Bose-Einstein condensate.

3. [50 pts.] Consider a 2D ideal boson system in area A whose chemical potential is μ and temperature is $T = \frac{1}{\beta}$. A boson particle in a state $|\vec{k}\rangle$ has energy $\epsilon_{\vec{k}} = \frac{\hbar^2 k^2}{2m}$.

- (a) Calculate the grand partition function Z_G and show that the average number of particle $n_{\vec{k}}$ in a state \vec{k} is given by

$$n_{\vec{k}} = \frac{1}{e^{\beta(\epsilon_{\vec{k}} - \mu)} - 1}$$

- (b) Represent the average particle number in the ground state, N_0 in terms of fugacity $z = e^{\beta\mu}$.

- (c) In 2D, density of state $g_1(\vec{k})$ for an ideal (spinless) boson system in the \vec{k} space is given by $g_1(\vec{k}) = \frac{A}{(2\pi)^2}$. Show that the density of state $g(\epsilon)$ in the energy space ϵ is written as

$$g(\epsilon) = C_1 A$$

and find an ϵ independent constant C_1 .

- (d) Show that the average number of particles in all excited states, $N_{ex} = N - N_0$ can be written as

$$N_{ex} = C_2 \frac{A}{\lambda^2} g_n(z)$$

and find C_2 and n .

- (e) Show that Bose-Einstein condensation does not occur at any finite temperature in 2D.