Thermal and Statistical Physics 2 (Quiz 1)

Date: 2012. 3. 28.

Collage:	Department:	Student Number:	Name:

∘ Do all 3 problems.

∘ No calculators, books, and notes are permitted.

1. [60 pts.] Hamiltonian for a system consists of N monoatomic free particles in a 3D box is given by

$$H = \sum_i \frac{\vec{p}_i^2}{2m},$$

with all particles in the box with volume V. (Otherwise, it is infinite.)

(a) Show that the partition function $Z(T, V, N)$ can be written as

$$Z(T, V, N) = \frac{1}{N!}\left(\frac{V}{\lambda^3}\right)^N,$$

when the system is in equilibrium with a thermal reservoir with temperature T. Write λ in terms of T.

(b) Now, let us consider a case when particles can go in and out of the box freely. Then the system becomes equilibrium with the reservoir. Assume that the reservoir has chemical potential μ and temperature T. Prove that the grand partition function, Z_G, of the system can be written as

$$Z_G(T, V, \mu) = \exp\left(\zeta V/\lambda^3\right), \tag{1}$$

and find ζ in terms of μ and T.

(c) Show that the grand thermodynamic function,

$$\Phi = U - TS - \mu N \tag{2}$$

is given by

$$\Phi = -T \ln Z_G,$$

using the definition of entropy, $S = -\sum_x p_x \ln p_x$.

(d) First, derive Gibbs-Duhem relation,

$$U = TS - pV + \mu N, \tag{3}$$

and then show

$$\Phi = -pV \tag{4}$$

form Eq. (2) and 3).

(e) Calculate the grand thermodynamic function of ideal gas.

(f) Find the equation of state for ideal gas using

$$\langle N \rangle = \frac{\partial \ln Z_G}{\partial\left(\mu/T\right)}$$

and Eqs. (1) and (4).

2. [20 pts.] Hamiltonian for a 1D harmonic oscillator is given by

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2.$$

Let's consider an eigen value problem given by

$$\hat{H}|\Psi\rangle = E|\Psi\rangle. \tag{5}$$

Assume that the real space representation of the operators $\hat{p}$ and $\hat{x}$ are given by

$$\langle x|\hat{x} = x\,\langle x|$$

$$\langle x|\hat{p} = -i\hbar\frac{\partial}{\partial x}\langle x|.$$

Derive the differential equation for $\Psi(x) = \langle x|\Psi\rangle$ from Eq. (5).

3. [20 pts.]

(a) Show that $|\phi_k\rangle$ is an eigen state with the eigen value a_k of the operator A, given by

$$A = \sum_i a_i|\phi_i\rangle\langle\phi_i|.$$

(b) Assume that the density operator, ρ, of a system is given by

$$\rho = \sum_i p_i\,|\psi_i\rangle\langle\psi_i|.$$

Calculate the expectation value $\langle G\rangle = \mathrm{Tr}\,\rho G$ of an operator G and discuss its meaning briefly.