Thermal and Statistical Physics 2 (Midterm Exam)

Date: 04/23, 2012

Collage:	Department:	Student Number:	Name:

∘ No calculators, books, and notes are permitted.

∘ HW Set 3: Solve these problems at home (Due: 04/30).

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1. [100 pts.] Consider a system consists of N spinless particles in a 1D harmonic potential. Hamiltonian of this system is given by

$$H(\{q_i, p_i\}) = \sum_{i=1}^{N} \frac{1}{2}kq_i^2 + \frac{p_i^2}{2m}.$$

(a) Show that the partition function, $Z(N, T, \omega(m, k))$ is given by

$$Z(N, T, \omega) = \frac{1}{N!}\left(\frac{T}{\hbar\omega}\right)^N,$$

when the system is in equilibrium with a reservoir with temperature T. What is ω in terms of m and k?

(b) Now, assume that the system becomes open and in equilibrium with a reservoir with temperature T and chemical potential μ. Calculate the grand partition function $Z_G = Z_G(\omega, T, \mu)$.

(c) Calculate the average particle number, $N = N(\omega, T, \mu)$.

(d) Calculate the grand thermodynamic potential $\phi = \phi(\omega, T, \mu)$.

(e) Calculate the Helmholtz free energy F, using $F = \phi + \mu N$ and represent it in terms of N, T and ω. Is it consistent with $-T \ln Z(N, T, \omega)$ with Z in (a).

2. [80 pts.] Consider a pure state of two spin-1/2 particles, particle-A and particle-B, given by

$$|\Phi\rangle = \frac{1}{\sqrt{10}}\left[|\uparrow\uparrow\rangle + \sqrt{2}|\uparrow\downarrow\rangle + \sqrt{3}|\downarrow\uparrow\rangle + 2|\downarrow\downarrow\rangle\right].$$

Here, $|\updownarrow\updownarrow\rangle$ represents $|\updownarrow_z\rangle_A|\updownarrow_z\rangle_B$.

(a) Represent the density state, ρ_{AB} of the A+B system with the basis $\{|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle\}$.

(b) Represent the density state ρ_A of the particle A with the basis $\{|\uparrow_A\rangle, |\downarrow_A\rangle\}$.

(c) Represent the density state ρ_B of the particle B with the basis $\{|\uparrow_B\rangle, |\downarrow_B\rangle\}$.

(d) The z-component of the spin of a spin-1/2 particle can be represented as

$$s_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

with the basis given in (b). Find the expectation value, $\langle s_z^A\rangle$ of the z-component of spin-A.

3. [80 pts.] Consider a spin-1/2 particle. The inner products between the eigen states in two different directions, x and z are given by,

$$\begin{aligned}\langle\uparrow_x|\uparrow_z\rangle &= \langle\downarrow_x|\uparrow_z\rangle = 1/\sqrt{2}\\ \langle\uparrow_x|\downarrow_z\rangle &= -\langle\downarrow_x|\downarrow_z\rangle = 1/\sqrt{2}\end{aligned}$$

(a) Represent $|\uparrow_x\rangle$ and $|\downarrow_x\rangle$ using the basis set $\{|\uparrow_z\rangle, |\downarrow_z\rangle\}$.

(b) Represent the density state, ρ, which consists of 80% of $|\uparrow_x\rangle$ state and 20% of $|\downarrow_x\rangle$.

(c) Hamiltonian of the above particle is given by

$$H = -\vec{S}\cdot\vec{B} = -(\mu B)\,\sigma_x$$

when it is in the magnetic field $\vec{B} = B\hat{x}$. Here $\mu = \frac{e\hbar}{2mc}$ is the Bohr magneton and $\left(\sigma_x\right)_{\{|\updownarrow_z\rangle\}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Show that the density state ρ of the particle can be represented as

$$\rho = C_1\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + C_2(\tanh C_3)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad (1)$$

when it is in equilibrium with a reservoir with temperature T. Find C_1, C_2 and C_3.

[If needed, use $e^{a\sigma_x} = (\cosh a)\,\mathbb{I} + (\sinh a)\sigma_x$.]

(d) If ρ of Eq. (1) consists of 80% of $|\uparrow_x\rangle$ state and 20% of $|\downarrow_x\rangle$ as in (b), what is corresponding temperature T?

4. [40 pts.] Show that the partition function Z of a free particle in a 1D box with length L can be written as

$$Z = L/\lambda,$$

and represent λ in terms of h, m, T, using

$$\begin{aligned} Z &= Tr\, e^{-H/T} \\ &= \sum_k \langle k|e^{-H/T}|k\rangle, \end{aligned}$$

and

$$H\,|k\rangle = \frac{\hbar^2 k^2}{2m}\,|k\rangle.$$