

Collage:

Department:

Student Number:

Name:

- o No calculators, books, and notes are permitted.
- o HW Set 3: Solve these problems at home (Due: 04/30).

1. [100 pts.] Consider a system consists of N spinless particles in a 1D harmonic potential. Hamiltonian of this system is given by

$$H(\{q_i, p_i\}) = \sum_{i=1}^N \frac{1}{2} k q_i^2 + \frac{p_i^2}{2m}.$$

- (a) Show that the partition function, $Z(N, T, \omega(m, k))$ is given by

$$Z(N, T, \omega) = \frac{1}{N!} \left(\frac{T}{\hbar \omega} \right)^N,$$

when the system is in equilibrium with a reservoir with temperature T . What is ω in terms of m and k ?

- (b) Now, assume that the system becomes open and in equilibrium with a reservoir with temperature T and chemical potential μ . Calculate the grand partition function $Z_G = Z_G(\omega, T, \mu)$.

- (c) Calculate the average particle number, $N = N(\omega, T, \mu)$.

- (d) Calculate the grand thermodynamic potential $\phi = \phi(\omega, T, \mu)$.

- (e) Calculate the Helmholtz free energy F , using $F = \phi + \mu N$ and represent it in terms of N , T and ω . Is it consistent with $-T \ln Z(N, T, \omega)$ with Z in (a).

2. [80 pts.] Consider a pure state of two spin-1/2 particles, particle-A and particle-B, given by

$$|\Phi\rangle = \frac{1}{\sqrt{10}} \left[|\uparrow\uparrow\rangle + \sqrt{2}|\uparrow\downarrow\rangle + \sqrt{3}|\downarrow\uparrow\rangle + 2|\downarrow\downarrow\rangle \right].$$

Here, $|\uparrow\downarrow\rangle$ represents $|\downarrow_z\rangle_A |\downarrow_z\rangle_B$.

- (a) Represent the density state, ρ_{AB} of the A+B system with the basis $\{|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle\}$.

- (b) Represent the density state ρ_A of the particle A with the basis $\{|\uparrow_A\rangle, |\downarrow_A\rangle\}$.

- (c) Represent the density state ρ_B of the particle B with the basis $\{|\uparrow_B\rangle, |\downarrow_B\rangle\}$.

- (d) The z -component of the spin of a spin-1/2 particle can be represented as

$$s_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

with the basis given in (b). Find the expectation value, $\langle s_z^A \rangle$ of the z -component of spin-A.

3. [80 pts.] Consider a spin-1/2 particle. The inner products between the eigen states in two different directions, x and z are given by,

$$\begin{aligned} \langle \uparrow_x | \uparrow_z \rangle &= \langle \downarrow_x | \uparrow_z \rangle = 1/\sqrt{2} \\ \langle \uparrow_x | \downarrow_z \rangle &= -\langle \downarrow_x | \downarrow_z \rangle = 1/\sqrt{2} \end{aligned}$$

- (a) Represent $|\uparrow_x\rangle$ and $|\downarrow_x\rangle$ using the basis set $\{|\uparrow_z\rangle, |\downarrow_z\rangle\}$.

- (b) Represent the density state, ρ , which consists of 80% of $|\uparrow_x\rangle$ state and 20% of $|\downarrow_x\rangle$.

- (c) Hamiltonian of the above particle is given by

$$H = -\vec{S} \cdot \vec{B} = -(\mu B) \sigma_x$$

when it is in the magnetic field $\vec{B} = B\hat{x}$. Here $\mu = \frac{e\hbar}{2mc}$ is the Bohr magneton and $(\sigma_x)_{\{|\uparrow_z\rangle\}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Show that the density state ρ of the particle can be represented as

$$\rho = C_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + C_2 (\tanh C_3) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (1)$$

when it is in equilibrium with a reservoir with temperature T . Find C_1 , C_2 and C_3 .

[If needed, use $e^{a\sigma_x} = (\cosh a) \mathbb{I} + (\sinh a) \sigma_x$.]

- (d) If ρ of Eq. (1) consists of 80% of $|\uparrow_x\rangle$ state and 20% of $|\downarrow_x\rangle$ as in (b), what is corresponding temperature T ?

4. [40 pts.] Show that the partition function Z of a free particle in a 1D box with length L can be written as

$$Z = L/\lambda,$$

and represent λ in terms of h , m , T , using

$$\begin{aligned} Z &= \text{Tr} e^{-H/T} \\ &= \sum_k \langle k | e^{-H/T} | k \rangle, \end{aligned}$$

and

$$H |k\rangle = \frac{\hbar^2 k^2}{2m} |k\rangle.$$