Thermal and Statistical Physics 2 (Homework set 2)

Due: 3 p.m., 2012. 04. 16.

Collage:	Department:	Student Number:	Name:

1. Hamiltonian for a system consists of N mono-atomic free particles in a 3D box is given by

$$H = \sum_i \frac{\vec{p}_i^2}{2m},$$

with all particles in the box with volume V. (Otherwise, it is infinite.)

(a) Show that the partition function $Z(T, V, N)$ can be written as

$$Z(T, V, N) = \frac{1}{N!}\left(\frac{V}{\lambda^3}\right)^N,$$

when the system is in equilibrium with a thermal reservoir with temperature T. Write λ in terms of T.

(b) Now, let us consider a case when particles can go in and out of the box freely. Then the system becomes equilibrium with the reservoir. Assume that the reservoir has chemical potential μ and temperature T. Prove that the grand partition function, Z_G, of the system can be written as

$$Z_G(T, V, \mu) = \exp\left(\zeta V/\lambda^3\right), \qquad (1)$$

and find ζ in terms of μ and T.

(c) Show that the grand thermodynamic function,

$$\Phi = U - TS - \mu N \qquad (2)$$

is given by

$$\Phi = -T \ln Z_G,$$

using the definition of entropy, $S = -\sum_x p_x \ln p_x$.

(d) First, derive Gibbs-Duhem relation,

$$U = TS - pV + \mu N, \qquad (3)$$

and then show

$$\Phi = -pV \qquad (4)$$

form Eq. (2) and (3).

(e) Calculate the grand thermodynamic function of ideal gas.

(f) Find the equation of state for ideal gas using

$$\langle N\rangle = \frac{\partial \ln Z_G}{\partial(\mu/T)}$$

and Eqs. (1) and (4).

2.

(a) Show that $|\phi_k\rangle$ is an eigen state with the eigen value a_k of the operator A, given by

$$A = \sum_i a_i |\phi_i\rangle\langle\phi_i|.$$

(b) Assume that the density operator, ρ, of a system is given by

$$\rho = \sum_i p_i\, |\psi_i\rangle\langle\psi_i|.$$

Calculate the expectation value $\langle G\rangle = \mathrm{Tr}\,\rho G$ of an operator G and discuss its meaning briefly.

3. Consider a pure state of two spin-1/2 particles, particle-A and particle-B, given by

$$|\Phi\rangle = c_1|\uparrow\uparrow\rangle + c_2|\uparrow\downarrow\rangle + c_3|\downarrow\uparrow\rangle + c_4|\downarrow\downarrow\rangle.$$

Here, $|\updownarrow\updownarrow\rangle$ represents $|\updownarrow_z\rangle_A|\updownarrow_z\rangle_B$.

(a) Choose $c_1 \cdots c_4$ such that $|\Phi\rangle$ be a seperable pure state. (Avoid the example we used in the class).

(b) Choose $c_1 \cdots c_4$ such that $|\Phi\rangle$ be an entangled pure state. (Avoid the example we used in the class).

4.

(a) Write down the density state of spin-2 in your example of the problem 3.(a) and show that it is a pure state.

(b) Write down the density state of spin-2 in your example of the problem 3.(b) and show that it is a mixed state.

5. A spin-1/2 particle A and a spin-1 particle B are coupled. Thier spin state $|\psi\rangle$ is given by

$$|\psi\rangle = \frac{1}{2}|\uparrow +\rangle - \frac{1}{2}|\uparrow 0\rangle + \frac{1}{\sqrt{2}}|\downarrow -\rangle.$$

Here $|\uparrow\rangle$ and $|\downarrow\rangle$ represent (the z-component) up state and down state of the particel A respectively, and $|+\rangle$, $|0\rangle$, and $|-\rangle$ represent (the z-component) +1, 0, and -1 state of the particel B respectively.

(a) Represent the density state, ρ_{AB} of the A+B system with the basis $\{|\uparrow +\rangle, |\uparrow 0\rangle, |\uparrow -\rangle, |\downarrow +\rangle, |\downarrow 0\rangle, |\downarrow -\rangle\}$.

(b) Represent the density state ρ_A of the particle A with the basis $\{|\uparrow\rangle, |\downarrow\rangle\}$.

(c) Represent the density state ρ_B of the particle B with the basis $\{|+\rangle, |0\rangle, |-\rangle\}$.

(d) The z-component of the spin of a spin-1 particle can be represented as

$$s_z = \hbar\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix}$$

with the basis given in (c). Find the expectation value, $\langle s_z^B\rangle$ of the z-component of spin-B.