Analysis of Non-Stationary Financial Time Series by Hilbert-Huang Transform

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Outline

1. Introduction
2. Hilbert-Huang transformation
3. Analysis of KOSPI 200
4. Conclusions
Goal and Data

- The goal of this talk
  - Introduce a novel method for analysis of financial time series by Hilbert-Huang transformation (HHT)
  - Apply HHT procedure to the KOSPI 200 index

- KOSPI 200 Index
Introduction

- In financial time series analysis, one of the main issues is modelling and forecasting price or index of the financial instrument.
- Usually transformation of financial time series rather than its original scale is taken for describing its dynamics.
- Main reason for taking a proper transformation is to convert non-stationary process to stationary process and subsequently to utilize favourable mathematical and statistical properties of stationary processes.
Introduction (cont.)

- However, the assumptions of stationarity and linearity are sometimes hard to hold for some financial time series
- Empirical mode decomposition (EMD) and Hilbert spectrum invented by Huang et al. (1998, 1999) provides new approach to analyze non-stationary and non-linear signals. This procedure is termed Hilbert-Huang transform (HHT)
Hilbert-Huang Transformation

- HHT is consists of two procedures, EMD and Hilbert spectral analysis.
- EMD decomposes a signal into so-called intrinsic mode function (IMF) according to the levels of their local oscillation or frequency.
- EMD efficiently captures non-linear characteristics respect to amplitude and frequency modulation with local time scale.
- Once IMFs are obtained, Hilbert spectral analysis provides frequency information varying over time that is a key component of time-frequency analysis for non-stationary financial time series.
Introduction to EMD

What is Frequency?

- Oscillating and periodic patterns are repeated
- Local mean is zero and the signal is symmetric to its local mean
- One cycle of oscillation: sinusoidal function starting at 0 and ending at 0 with passing through zero between two zero crossings. Or starting at local maximum and terminating at consecutive local maximum with passing through two zeros and local minimum
Huang et. al (1998) defined an oscillating wave as **Implicit Mode Function (IMF)** if it satisfies two conditions

1) the number of extrema and the number of zero crossing differs only by one and

2) the local average is zero. The condition that the local average is zero implies that envelop mean of upper envelop and lower envelop is zero
Introduction to EMD (cont.)

- A signal observed in real world consist of low and high frequencies
- Suppose we observe signal $x(t)$ which is of the form

$$x(t) = 0.5t + \sin(\pi t) + \sin(2\pi t) + \sin(6\pi t) + \epsilon,$$

where $\epsilon$ is the noise

- For illustrative purpose, ignore noise this time. The signal consist of 4 components from the highest frequency $\sin(6\pi t)$ to the lowest frequency $0.5t$
Sifting – Extracting Implicit Mode Function (IMF)

1. First identify the local extrema
2. Consider the two functions interpolated by local maximum and local minimum (upper and lower envelop)
3. Their average, envelop mean will yields the lower frequency component than the original signal
4. By subtracting envelop mean, from the original signal $x(t)$, the highly oscillated pattern $h$ is separated
Sifting – Extracting Implicit Mode Function (IMF) (cont.)
Sifting – Extracting Implicit Mode Function (IMF) (cont.)

- One iteration of above procedure does not guarantee that the resulting residue signal $h$ is IMF. The same procedure is applied to the residue signal $h$ until properties of IMF is satisfied.
- This iterative algorithm is called sifting.
- Sifting makes the remaining signal more symmetric, local mean toward to zero, all the maximum positive and all the minimum negative.
- In other word, sifting makes envelop mean to bisect the signal evenly so that there are no overshoots nor undershoots.
- It is known that cubic spline is optimal for interpolating.
Sifting – Extracting Implicit Mode Function (IMF) (cont.)
Sifting – Extracting Implicit Mode Function (IMF) (cont.)

- Note that as the name sifting implies, the lower frequency component is repeatedly removed from the highest frequency.
- The first IMF $imf_1$ produced by sifting is the highest frequency by its construction.
- Residue signal $r$ less oscillated than the original signal. Remaining signal $r = x - imf_1$ still may be compound of several frequencies.
- The same procedure is applied on the residue signal $r$ to obtain the next IMF.
- By the construction, the number of extrema will eventually decreased as the procedure continues so that a signal is sequently decomposed into the highest frequency component $imf_1$ to the lowest frequency component $imf_n$, for some finite $n$ and residue $r$.

Finally we have $n$ empirical mode and residue as

$$x(t) = \sum_{i=1}^{n} imf_i(t) + r(t)$$
Sifting – Extracting Implicit Mode Function (IMF) (cont.)

Signal = 1–st IMF + 1–st residue

1–st residue = 2–nd IMF + 2–nd residue

1–st imf

2–nd imf

1–st residue

2–nd residue
Sifting – Extracting Implicit Mode Function (IMF) (cont.)
Sifting – Extracting Implicit Mode Function (IMF) (cont.)

EMD Algorithm

a. Take input signal $r_{k-1}$ to decompose. $r_0$ is the original signal $x$
   1. Identify the local extrema of the signal $r_{k-1}$
   2. Construct upper envelop $emax_k$ and lower envelop $emin_k$ interpolating maximum and minimum, respectively
   3. Approximate local average by envelop mean $em_k$ taking average of two envelops $emax_k$ and $emin_k$. That is $em_k = (emax_k + emin_k)/2$
   4. Compute candidate implicit mode $h_{k1} = r_{k-1} - em_k$
   5. If $h_{ki}$ is IMF, decompose the signal $r_{k-1}$ as IMF $imf_k = h_{ik}$ and the residue signal $r_k = r_{k-1} - imf_k$. Otherwise repeat the step 1 through 5

b. If $r_k$ has implicit oscillation mode, set $r_k$ as input signal and repeat step a
Hilbert Transform and Instantaneous Frequency

Traditional spectral analysis based on Fourier analysis would be meaningless for non-stationary time series for which the frequency and amplitude are changing over time.

- For a real signal $X(t)$, analytic signal $Z(t)$ is defined as $Z(t) = X(t) + iY(t)$ where $Y(t)$ is the Hilbert transform of $X(t)$, that is,

$$Y(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{X(s)}{t-s} ds,$$

where $P$ is the Cauchy principle value.
Hilbert Transform and Instantaneous Frequency (cont.)

- The analytic signal \( Z(t) \) can be represented by a polar coordinate form with amplitude and phase as:

\[
Z(t) = a(t) \exp(i\theta(t)),
\]

where \( a(t) = \|Z(t)\| = \sqrt{X(t)^2 + Y(t)^2} \) and \( \theta(t) = \arctan(Y(t)/X(t)) \)

- The analytic signal can capture the local characteristics of a signal \( X(t) \) since 1) the Hilbert transform is the convolution of with \( 1/t \), and 2) the polar coordinate form provides time-varying amplitude and phase

- Define instantaneous frequency as the derivative of time-varying phase,

\[
\frac{d\theta(t)}{dt}
\]
Hilbert-Huang Transformation

- Once EMD decomposes the signal into IMF’s, apply the Hilbert transform to the IMF’s and perform the Hilbert spectrum to obtain amplitude and instantaneous frequency representation with respect to time.

- Through the Hilbert transform, the IMF’s yield instantaneous frequencies as a function of time which identify hidden local structures embedded in original signal.

- HHT provides us useful tool in that any local property can be preserved on the time domain (by EMD) as well as on the frequency domain (by Hilbert transform).
Hilbert-Huang Transformation (cont.)

It shows Hilbert spectrum for IMF1 and IMF2 of the signal. The X-Y axis represents time and instantaneous frequency, and the gray intensity of the image depicts instantaneous amplitude.
Analysis of KOSPI 200

The ultimate goal of the talk is to investigate weekly KOSPI 200 index from January, 1990 to February, 2007 to demonstrate applicability of Hilbert-Huang transform.

- The behaviour of the index illustrates non-stationary features
- Observe three major waves from the mid 1992 to the mid 1998, from the mid 1998 to the mid 2001, and from the mid 2001 to the mid 2003
- Fluctuation is severe and the volatility is increasing between year 1997, Asian financial crisis and year 2001
Decomposition of KOSPI 200

- EMD decomposes this index into eight IMF’s and a global trend.
- IMF1 and IMF2 represent high frequency characteristic of the index, while IMF4 through IMF6 extract mid-range frequency signals.
- Especially, IMF4 through IMF6 correspond to the components with period 6 month, 1 year and 2 year, respectively.
- Long-term behaviours are well described through IMF7 and IMF8. The IMF7 detects 5 year cycle dynamics from 1993 to 1998 and two waves from 1998 to the mid 2003. By combining IMFs 7 and 8, the pattern of the three major waves is well replicated.
Decomposition of KOSPI 200 (cont.)
EMD, Filter and Denoising

- IMF’s of high frequencies contain localized information at a specific time, and IMF’s of low frequencies describe trend over the whole time span.
- EMD can play a role as a filter by properly choosing the resolution level of the IMF’s.
- A pre-processing is required to smooth out the noise from the observations.
- To extract noise from signal, define low pass filter

\[ L_k(t) = \sum_{i=k}^{n} imf_i(t) + r(t). \]

- By controlling the amount of local information of each IMF’s, i.e. choosing proper level \( k \) in \( L_k(t) \), EMD smooths out the noisy in the signal.
EMD, Filter and Denoising (cont.)
Hierarchical Smoothing Technique by Cross-Validation and EMD

- The procedure can be summarized as follows: for properly selected threshold values $\lambda_i \ i = 1, \ldots, k$, the thresholded IMF’s $d_1, \ldots, d_k$ are defined as
  \[
  d_i(t) = \begin{cases} 
  0 & \text{if } |imf_i(t)| < \lambda_i \\
  imf_i(t) & \text{otherwise}. 
  \end{cases}
  \]

- By recombining the thresholded IMF’s and residue signal, a denoised signal $\hat{X}(t)$ is constructed as
  \[
  \hat{X}(t) = \sum_{i=1}^{k} d_i(t) + \sum_{i=k+1}^{L} imf_i(t) + r(t)
  \]

which can be considered as a generalization of the low pass filter
Hierarchical Smoothing Technique by Cross-Validation and EMD (cont.)

- For thresholding, we need a criterion to select $\lambda_i$
- To choose thresholding values, we use the following level-dependent $M$-fold cross-validation

$$CV(\lambda_1, \ldots, \lambda_k) = \frac{1}{n} \sum_{t=1}^{n} \{ X(t) - \hat{X}(t)_{\lambda_1, \ldots, \lambda_k}(t) \}^2,$$

where $\hat{X}(t)_{\lambda_1, \ldots, \lambda_k}(t)$ indicates, given a threshold $\lambda_i$ for the $i$th IMF, the estimate of $X(t)$ computed by expelling the $m(t)$th part of signal

- Then find the thresholds $\lambda_1, \ldots, \lambda_k$ that minimizes $CV(\lambda_1, \ldots, \lambda_k)$ over a suitable range
Hierarchical Smoothing Technique by Cross-Validation and EMD (cont.)
Fluctuation Measure of KOSPI 200

- IMF’s with high frequency are source of the volatility of the KOSPI 200 index and the volatility itself is volatile according to time.

- To measure the volatility of the signal according to time, EMD can be utilized as a high pass filter. The high pass filter $H_k(t)$ is

$$H_k(t) = \sum_{i=1}^{k} imf_i(t), \text{ for some } k.$$

- Note that our intention is not to estimate the volatility as a model parameter, but to build an indicator for fluctuation according to time.

- To identify a volatile signal at a particular time, define a fluctuation measure based on $H_k(t)$ as

$$V(H_k) = \sum_{i=1}^{k} imf_i^2(t), \text{ for some } k.$$
Fluctuation Measure of KOSPI 200 (cont.)
Fluctuation Measure of KOSPI 200 (cont.)

- Hilbert spectrum $HS$ measures the local energy, i.e. instantaneous amplitude and frequency in $(t, \omega)$ dimension, the time-frequency dimension

- The squared energy $E_k$ up to the $k$th IMF might be an alternative for indicator of the volatility

$$E_k = \sum_{imf_i, \ldots, imf_k} HS^2(t, \omega).$$
Fluctuation Measure of KOSPI 200 (cont.)
Fluctuation Measure of KOSPI 200 (cont.)
Conclusions

- This talk introduces HHT, the two-step procedure combining EMD and Hilbert spectrum.
- HHT is a data-adaptive method capturing local properties, easy to implement and robust to presence of non-linearity and non-stationarity.
- We demonstrate the promising capability of HHT for non-stationary financial time series data through KOSPI 200 index.
- Applications of HHT include decomposition of complicated signal, denoising, detecting volatility and etc.
- HHT provides a new viewpoint in dealing with non-stationary financial time series and broaden the scope for real applications.