Technical Note

A note on linear dispersion and shoaling properties in extended Boussinesq equations

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Abstract

A set of optimum parameter $\alpha$ is obtained to evaluate the linear dispersion and shoaling properties in the extended Boussinesq equations of Madsen and Sørensen (1992), Nwogu (1993), and Chen and Liu (1995). Optimum $\alpha$ values are determined to produce minimal errors in each wave property of phase velocity, group velocity, or shoaling coefficient relative to the analytical one given by the Stokes wave theory. Comparisons are made of the percent errors in phase velocity, group velocity, and shoaling coefficient produced by the Boussinesq equations with a different set of optimum $\alpha$ values. The case with a fixed value of $\alpha = -0.4$ is also presented in the comparison. The comparisons reveal that the optimum $\alpha$ value tuned for a particular wave property gives in general poor results for other properties. Considering all the properties simultaneously, the fixed value of $\alpha = -0.4$ may give overall accuracies in phase velocity and shoaling coefficient for all the types of Boussinesq equations selected in this study.

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1. Introduction

The propagation of water waves from ocean to coastal area is a dynamic phenomenon which can bring human beings happiness some times or disaster in other times.
Identification of physical behavior of water wave propagation is an ongoing research topic. Among the mathematical models, the Boussinesq equations are known to predict the propagation of random and nonlinear water waves with high accuracy especially in shallow water. The Boussinesq equations, which are composed of the continuity equation and the x- and y-momentum equations, were derived based on the two small parameters, i.e., the relative water depth $kh$ ($k$ is the wavenumber and $h$ is the water depth) representing dispersiveness of waves and the relative wave amplitude $a/h$ ($a$ is the wave amplitude) measuring the nonlinearity of waves, and on the assumption of $(k/h)^2 = O(a/h)$. However, the accuracy of Boussinesq equations decreases as the water becomes deep.

Recently, the problem involved in the application of Boussinesq equations to deeper water has been partially solved by improving their linear dispersion relations. Witting (1984) is the first who showed the possibility of the extension of Boussinesq equations for deeper water. He used the Padé approximation in connection with the linear Stokes wave theory. Using Padé approximants in terms of $kh$ to order (N,N) gives high accuracy to order 2N without increasing the order of the derivatives. This gives excellent dispersion and shoaling properties. However, it is difficult to extend the approach of Witting directly to two horizontal dimensions with variable water depth. The first type of extension of the Boussinesq equations has been made by adding some correction terms with a tuning parameter $B$ to the momentum equations so that the reduced linear dispersion relations match the Padé [2,2] approximation to the dispersion relations given by the linear Stokes wave theory (Madsen and Sørensen, 1992). The second type of extension has been made by using, instead of depth-averaged velocities, horizontal velocities on an arbitrary elevation with a tuning parameter $\alpha$ so that the reduced linear dispersion relations are approximate to those given by the linear Stokes wave theory (Nwogu, 1993). The third type of extension has been made, similarly to the second type, by using the velocity potential on an arbitrary elevation with a tuning parameter $\alpha$ (Chen and Liu, 1995). The three types of extended Boussinesq equations have been extended further by considering the higher order in nonlinearity (Wei et al., 1995).

The linear dispersion relations of various extended Boussinesq equations depend on the choice of the tuning parameter $B$ or $\alpha$. Comparison of the linear dispersion relations of extended Boussinesq equations reveals the relation of $B = -(\alpha + 1/3)$.

The choice of $\alpha = -1/3$ yields the dispersion relation of the conventional Boussinesq equations of Peregrine (1967). The choice of $\alpha = -0.4$, i.e., $B = 1/15$ yields that of the Boussinesq equations of Madsen and Sørensen (1992). The choice of $\alpha = -0.3900$ yields the minimal squared relative errors in phase velocity in the depth range of $0 < k_0h < \pi$ (Nwogu, 1993) where $k_0$ is the wavenumber in deep water. And the choice of $\alpha = -0.3855$ yields the minimal sum of squared relative errors in phase and group velocities in the depth range of $0 < k_0h < \pi$ (Chen and Liu, 1995).

In all the extended Boussinesq equations developed so far, the values of tuning parameter are fixed regardless of the water depth. Thus, the errors in the dispersion relations vary with different water depth. There exist still non-zero errors in the reduced linear dispersion relations in both intermediate-depth and deep waters, and
the magnitude of errors increases as the water becomes deeper. Water waves transform in both phase and energy due to shoaling, refraction, diffraction, and reflection. The degree of refraction, diffraction, and reflection depends on the dispersion relation, and the degree of shoaling depends on the shoaling coefficient.

In this study, the values of the tuning parameter are chosen to vary with local water depth so that squared relative errors in linear dispersion relation or shoaling coefficient may be minimal in each water depth. Three types of the extended Boussinesq equations by Madsen and Sørensen, Nwogu, and Chen and Liu are chosen in this study. In section 2, the linear dispersion relations and shoaling equations for three types of the Boussinesq equations are derived using the geometric optics approach. In section 3, the accuracy of linear dispersion relations and shoaling properties associated with various extended Boussinesq equations with a fixed $\alpha$ is evaluated. In section 4, the optimum values of tuning parameter $\alpha$ for each set of Boussinesq equations are determined locally to minimize squared relative errors in the various wave properties of phase velocity, group velocity, or shoaling coefficient. Then, comparisons are made of the relative errors in wave properties obtained with the optimum parameter. The errors with the fixed value of $\alpha = -0.4$ are also compared. In section 5, in order to verify the analytical predictions, numerical experiments are conducted in a horizontally one-dimensional domain using the linearized equations of Nwogu. Finally, concluding remarks are presented.

2. Derivation of linear dispersion relation and shoaling equation

Following the approach of Madsen and Sørensen (1992) the linear dispersion relation and shoaling equation can be derived for the various extended Boussinesq equations of Madsen and Sørensen (1992), Nwogu (1993), and Chen and Liu (1995). To compare three types of the extended Boussinesq equations at the same time, we replace the parameter $B$ as $-(\alpha + 1/3)$ in the equations of Madsen and Sørensen.

In the case that the domain is horizontally one-dimensional and nonlinear terms are neglected, the equations of Madsen and Sørensen are reduced to

$$\frac{\partial \eta}{\partial t} + \frac{\partial P}{\partial x} = 0$$

$$\frac{\partial P}{\partial t} + gh\frac{\partial \eta}{\partial x} + \alpha h^2 \frac{\partial^3 P}{\partial x^3} + \left(\alpha + \frac{1}{3}\right)gh^3 \frac{\partial^3 \eta}{\partial x^3} - h^2 \frac{\partial h}{\partial x} \left[\frac{1}{3} \frac{\partial^2 P}{\partial x \partial t}\right]$$

$$-2\left(\alpha + \frac{1}{3}\right)gh \frac{\partial^2 \eta}{\partial x^2} = 0,$$

the equations of Nwogu are reduced to

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x} (hu) + \frac{\partial}{\partial x}\left[\left(\frac{z_\alpha}{2} - \frac{h^2}{6}\right) h^2 \frac{\partial^2 u}{\partial x^2} + \left(z_\alpha + \frac{h}{2}\right) h^2 \frac{\partial^2}{\partial x^2} (hu)\right] = 0$$
\begin{align}
\frac{\partial u}{\partial t} + g \frac{\partial \eta}{\partial x} + \frac{\zeta^2}{2} \frac{\partial^3 u}{\partial x^3 \partial t} + z^2 \frac{\partial^2}{\partial x^2} \left( \frac{\partial u}{\partial t} \right) &= 0, \\
\text{(4)}
\end{align}

and the equations of Chen and Liu are reduced to
\begin{align}
\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x} \left( \frac{\partial \phi}{\partial x} \right) + \frac{\partial}{\partial x} \left[ \frac{\partial}{\partial x} \left( \frac{\partial \phi}{\partial x} \right) \right] + \frac{\partial^2}{\partial x^2} \left( \frac{\partial \phi}{\partial x} \right) &= 0, \\
- \frac{h^3}{6} \frac{\partial^2 \phi}{\partial x^2} &= 0 \\
\frac{\partial \phi}{\partial t} + g \eta + \frac{\zeta^2}{2} \frac{\partial^3 \phi}{\partial x^3} + z^2 \frac{\partial}{\partial x} \left( h \frac{\partial^2 \phi}{\partial x^2} \right) &= 0 \\
\text{(5)}
\end{align}

In the above equations \( \eta \) is the water surface elevation, \( P \) is the volume flux in \( x \)-direction, \( u \) is the particle velocity in \( x \)-direction at \( z = z_\alpha \), \( \phi \) is the velocity potential at \( z = z_\alpha \), \( g \) is the acceleration due to gravity, and \( \alpha = (z_\alpha/h)^2/2 + z_\alpha/h \).

From the Madsen and Sørensen’s Eqs. (1) and (2), the following equation can be obtained in terms of the surface elevation \( \eta \) as
\begin{align}
\frac{\partial^2 \eta}{\partial t^2} - g \frac{\partial}{\partial x} \left( \frac{\partial \eta}{\partial x} \right) + \frac{\partial}{\partial x} \left[ \frac{\partial^2 \eta}{\partial x^2} \right] - \frac{\partial}{\partial x} \left[ \alpha \frac{\partial^2 \eta}{\partial x^2} \right] - \frac{\partial}{\partial x} \left[ \frac{h^2}{\partial x^2} \right] - \frac{\partial}{\partial x} \left[ \frac{3}{\partial x^3} \right] &= 0, \\
+ 2g \left( \alpha + \frac{1}{3} \right) \frac{\partial^3 \eta}{\partial x^3} &= 0 \\
\text{(7)}
\end{align}

The surface elevation can be defined as
\begin{align}
\eta = a e^{i\psi} \\
\text{(8)}
\end{align}

where \( a \) is the amplitude of water surface elevation, \( \psi \) is the phase function which has the following relation with the wave number \( k \) and the angular frequency \( \omega \):
\begin{align}
k = \frac{\partial \psi}{\partial x}, \quad \omega = -\frac{\partial \psi}{\partial t} \\
\text{(9)}
\end{align}

After substituting Eq. (8) into Eq. (7), rearranging terms of \( O(1) \) yields the following linear dispersion relation:
\begin{align}
C^2 = \left( \frac{\omega}{k} \right)^2 &= gh \frac{1 - \left( \alpha + \frac{1}{3} \right) k^2 h^2}{1 - \alpha k^2 h^2} \\
\text{(10)}
\end{align}

and rearranging terms of \( O(\partial a/\partial x, \partial k/\partial x, \partial h/\partial x) \) yields the following equation
\begin{align}
\alpha_1 \frac{\partial a}{\partial x} + \alpha_2 \frac{\partial k}{\partial x} + \alpha_3 \frac{\partial h}{\partial x} &= 0 \\
\text{(11)}
\end{align}

where the coefficients \( \alpha_1, \alpha_2, \alpha_3 \) are
\[
\alpha_1 = 2 \left[ 1 - 2 \left( \alpha + \frac{1}{3} \right) k^2 h^2 + \alpha \left( \alpha + \frac{1}{3} \right) k^4 h^4 \right]
\] (12)

\[
\alpha_2 = 1 - 6 \left( \alpha + \frac{1}{3} \right) k^2 h^2 + 5 \alpha \left( \alpha + \frac{1}{3} \right) k^4 h^4
\] (13)

\[
\alpha_3 = 1 - 2(2 \alpha + 1) k^2 h^2 + \left( \alpha + \frac{1}{3} \right) \left( 3 \alpha + \frac{1}{3} \right) k^4 h^4
\] (14)

Taking derivative of Eq. (10) with respect to \(x\) gives

\[
\frac{1}{k} \frac{\partial k}{\partial x} = -\alpha_4 \frac{\partial h}{h \partial x}
\] (15)

where the coefficient \(\alpha_4\) is given by

\[
\alpha_4 = \frac{1 - (2 \alpha + 1) k^2 h^2 + \alpha \left( \alpha + \frac{1}{3} \right) k^4 h^4}{2 \left[ 1 - 2 \left( \alpha + \frac{1}{3} \right) k^2 h^2 + \alpha \left( \alpha + \frac{1}{3} \right) k^4 h^4 \right]}
\] (16)

Substituting Eq. (15) into Eq. (11) gives the following shoaling equation:

\[
\frac{1}{a} \frac{\partial a}{\partial x} = -\alpha_5 \frac{\partial h}{h \partial x}
\] (17)

where the coefficient \(\alpha_5\) is given by

\[
\alpha_5 = \frac{\alpha_3 - \alpha_2 \alpha_4}{\alpha_1}
\] (18)

Using the same approach presented by Madsen and Sørensen, the linear dispersion relation and shoaling equation can be obtained for the equations of Nwogu. From the Nwogu’s Eqs. (3) and (4), the following equation can be reduced in terms of \(u\) as

\[
\frac{\partial^2 u}{\partial t^2} - g \frac{\partial^2}{\partial x^2} \left( hu \right) - g \frac{\partial^2}{\partial x^2} \left[ \left( \frac{z^2}{2} - \frac{h^2}{6} \right) \frac{\partial^2 u}{\partial x^2} + \left( z_\alpha + \frac{h}{2} \right) \frac{\partial^2 u}{\partial x^2} \left( hu \right) \right] + \frac{z_\alpha^2}{2} \frac{\partial^4 u}{\partial x^2 \partial t^2}
\] (19)

\[+ z_\alpha \frac{\partial^2}{\partial x^2} \left( \frac{\partial^2 u}{\partial t^2} \right) = 0\]

The particle velocity \(u\) can be defined as

\[u = A_u e^{i\omega t}\]
(20)

where \(A_u\) is the amplitude of particle velocity \(u\). After substituting Eq. (20) into Eq. (19), rearranging terms of \(O(1)\) yields the dispersion relation Eq. (10) which is the same as for Madsen and Sørensen’s Boussinesq equations. Rearranging terms of \(O(\partial A_u/\partial x, \partial k/\partial x, \partial h/\partial x)\) yields
\[ \beta_1 \frac{1}{A_u} \frac{\partial A_u}{\partial x} + \beta_2 \frac{1}{k} \frac{\partial k}{\partial x} + \beta_3 \frac{1}{h} \frac{\partial h}{\partial x} = 0 \]  

(21)

where the coefficients \( \beta_1, \beta_2, \beta_3 \) are

\[ \beta_1 = 2 - 4 \left( \alpha + \frac{1}{3} \right) k^2 h^2 + \frac{2\alpha \left[ 1 - \left( \alpha + \frac{1}{3} \right) k^2 h^2 \right]}{1 - \alpha k^2 h^2} k^2 h^2 \]  

(22)

\[ \beta_2 = 1 - 6 \left( \alpha + \frac{1}{3} \right) k^2 h^2 + \frac{\alpha \left[ 1 - \left( \alpha + \frac{1}{3} \right) k^2 h^2 \right]}{1 - \alpha k^2 h^2} k^2 h^2 \]  

(23)

\[ \beta_3 = 2 - 2 \left[ \frac{3(\alpha^2)}{2} \right]^2 + 4 \frac{\alpha}{h} + \frac{3}{2} \right] k^2 h^2 + \frac{1 - \left( \alpha + \frac{1}{3} \right) k^2 h^2}{1 - \alpha k^2 h^2} k^2 h^2 \]  

(24)

It is needed to get a shoaling equation in terms of the amplitude of surface elevation rather than particle velocity. Substituting Eqs. (8) and (20) into Eq. (4) and differentiating the resulting equation with respect to \( x \) yield the following equation:

\[ \frac{1}{A_u} \frac{\partial A_u}{\partial x} = \frac{1}{A_u} \frac{\partial a}{\partial x} + \left( 1 + \frac{2\alpha k^2 h^2}{1 - \alpha k^2 h^2} \right) \frac{1}{k} \frac{\partial k}{\partial x} + \frac{2\alpha k^2 h^2}{1 - \alpha k^2 h^2} \frac{1}{h} \frac{\partial h}{\partial x} \]  

(25)

where the terms higher than \( O(\partial a/\partial x, \partial A_u/\partial x, \partial k/\partial x, \partial h/\partial x) \) are neglected. Substituting Eq. (25) into Eq. (21) yields Eq. (11) but with the coefficients \( \alpha_1, \alpha_2, \alpha_3 \) given by

\[ \alpha_1 = 2 + \frac{2k^2 h^2}{1 - \alpha k^2 h^2} \left[ -\alpha - \frac{2}{3} + \alpha \left( \alpha + \frac{1}{3} \right) k^2 h^2 \right] \]  

(26)

\[ \alpha_2 = 3 + \frac{k^2 h^2}{[1 - \alpha k^2 h^2]^2} \left[ -3\alpha - \frac{10}{3} + 3\alpha(2\alpha + 1)k^2 h^2 \right] \]  

(27)

\[ -3\alpha \left( \alpha + \frac{1}{3} \right) k^4 h^4 \]

\[ \alpha_3 = 2 + \frac{2k^2 h^2}{1 - \alpha k^2 h^2} \left[ -\alpha - \frac{3}{2} + \left[ \alpha \left( \alpha + \frac{5}{6} \right) - \frac{1}{3} \frac{z_a}{h} - \frac{2\alpha}{3[1 - \alpha k^2 h^2]} \right] k^2 h^2 \right] \]  

(28)

Substituting Eq. (15) into Eq. (11) yields the shoaling Eq. (17). It should be noted that the shoaling equation for Nwogu’s equations is different from that for Madsen and Sørensen’s equations, because the coefficients \( \alpha_1, \alpha_2, \alpha_3 \) are different between these two equations.

Using the same approach proposed by Madsen and Sørensen, the linear dispersion relation and shoaling equation can be obtained for the equations of Chen and Liu. From Chen and Liu’s Eqs. (5) and (6), the following equation can be obtained in terms of \( \phi \) as
\[
\frac{\partial^2 \phi}{\partial t^2} - g \frac{\partial}{\partial x} \left( \frac{\partial \phi}{\partial x} \right) - g \frac{\partial}{\partial x} \left[ \frac{z \partial}{\partial x} \left( \frac{\partial \phi}{\partial x} \right) + \frac{z^2}{2} \frac{\partial^2 \phi}{\partial x^2} \right] + \frac{h^2}{2} \frac{\partial^2}{\partial x^2} \left( \frac{\partial \phi}{\partial x} \right) = 0
\]

(29)

\[
\frac{\partial^3 \phi}{\partial x^3} + \frac{z^2}{2} \frac{\partial^4 \phi}{\partial x^4} + \frac{z}{\alpha} \frac{\partial}{\partial x} \left( \frac{\partial^3 \phi}{\partial x \partial t^2} \right) = 0
\]

The velocity potential \( \phi \) can be defined as

\[
\phi = A_\phi e^{iy}
\]

(30)

where \( A_\phi \) is the amplitude of velocity potential. After substituting Eq. (30) into Eq. (29), rearranging terms of \( O(1) \) yields the dispersion relation Eq. (10) which is the same as for Madsen and Sørensen’s Boussinesq equations. As a result, all the three extended Boussinesq equations have the same dispersion relation in terms of the parameter \( B \) of Madsen and Sørensen can be replaced by \( - (\alpha + 1/3) \). Rearranging terms of \( O(\partial A_\phi/\partial x, \partial k/\partial x, \partial h/\partial x) \) yields

\[
\gamma_1 \frac{\partial A_\phi}{\partial x} + \gamma_2 \frac{\partial k}{\partial x} + \gamma_3 \frac{\partial h}{\partial x} = 0
\]

(31)

where the coefficients \( \gamma_1, \gamma_2, \gamma_3 \) are

\[
\gamma_1 = 2 - 4 \left( \alpha + \frac{1}{3} \right) k^2 h^2 + \frac{2\alpha \left[ 1 - \left( \alpha + \frac{1}{3} \right) k^2 h^2 \right]}{1 - \alpha k^2 h^2} k^2 h^2
\]

(32)

\[
\gamma_2 = 1 - 6 \left( \alpha + \frac{1}{3} \right) k^2 h^2 + \frac{\alpha \left[ 1 - \left( 1 + \frac{1}{3} \right) k^2 h^2 \right]}{1 - \alpha k^2 h^2} k^2 h^2
\]

(33)

\[
\gamma_3 = 1 - \left[ \frac{5}{2} \left( \frac{z}{h} \right)^2 + 6 \frac{z}{h} + 2 \right] k^2 h^2 + \frac{z}{h} \left[ 1 - \left( \alpha + \frac{1}{3} \right) k^2 h^2 \right] k^2 h^2
\]

(34)

The coefficients of \( \beta_1 \) and \( \gamma_1 \) have the relation of \( \beta_1 = \gamma_1 \), \( \beta_2 = \gamma_2 \), and \( \beta_3 \neq \gamma_3 \).

It is needed to get a shoaling equation in terms of the amplitude of surface elevation rather than velocity potential. Substituting Eqs. (8) and (30) into Eq. (6) and differentiating the resulting equation with respect to \( x \) yield the following equation:

\[
\frac{1}{A_\phi} \frac{\partial A_\phi}{\partial x} = \frac{1}{a} \frac{\partial a}{\partial x} + \frac{2\alpha k^2 h^2}{1 - \alpha k^2 h^2} \left( \frac{1}{k} \frac{\partial k}{\partial x} + \frac{1}{h} \frac{\partial h}{\partial x} \right)
\]

(35)

where the terms higher than \( O(\partial a/\partial x, \partial A_\phi/\partial x, \partial k/\partial x, \partial h/\partial x) \) are neglected. Substituting Eq. (35) into Eq. (31) yields Eq. (11) but with the coefficients \( \alpha_1, \alpha_2, \alpha_3 \) given by

\[
\alpha_1 = 2 + \frac{2k^2 h^2}{1 - \alpha k^2 h^2} \left[ -\alpha - 2 + \alpha \left( \alpha + \frac{1}{3} \right) k^2 h^2 \right]
\]

(36)

\[
\alpha_2 = 1 + \frac{k^2 h^2}{(1 - \alpha k^2 h^2)^2} \left[ -\alpha - 2 + \alpha(2\alpha + 1) k^2 h^2 - \alpha^3 \left( \alpha + \frac{1}{3} \right) k^4 h^4 \right]
\]

(37)
Substituting Eq. (15) into Eq. (11) yields the shoaling Eq. (17). Chen and Liu (1995) derived the shoaling equation where the coefficients are different from those in the present study. However, the two shoaling equations are found to be equivalent to each other.

The linear dispersion relation and shoaling equation for extended Boussinesq equations can be compared against those for the linear Stokes wave. The linear dispersion relation for the Stokes wave is given by

$$\alpha_3 = 1 + \frac{k^2h^2}{1-\alpha k^2h^2}\left(-\alpha - 2 + \left[\alpha + 2 + \frac{2}{3}\frac{z_\alpha}{h} - \frac{4\alpha}{3(1-\alpha k^2h^2)}\right]k^2h^2\right)$$  \hspace{1cm} (38)

Substituting Eq. (15) into Eq. (11) yields the shoaling Eq. (17). Chen and Liu (1995) derived the shoaling equation where the coefficients are different from those in the present study. However, the two shoaling equations are found to be equivalent to each other.

The linear dispersion relation and shoaling equation for extended Boussinesq equations can be compared against those for the linear Stokes wave. The linear dispersion relation for the Stokes wave is given by

$$C_l^2 = \left(\frac{\omega}{k_l}\right)^2 = \frac{g}{k_l}\tanh k_l h$$  \hspace{1cm} (39)

where the subscript \(l\) denotes the value for the linear Stokes wave. The coefficient \(\alpha_5\) of the shoaling equation for the linear Stokes wave is (Madsen and Sørensen, 1992)

$$\alpha_5 = \frac{2k_l h}{\sinh 2k_l h}\left[1 + \frac{k_l h}{\sinh 2k_l h}(1-\cosh 2k_l h)\right]$$  \hspace{1cm} (40)

It is needed to evaluate the group velocity in the extended Boussinesq equations. The group velocity is the velocity of wave energy which is shown in the transport equation for wave energy given by

$$\frac{\partial a^2}{\partial t} + \frac{\partial}{\partial x}(C_g a^2) = 0$$  \hspace{1cm} (41)

In the case of depth variation the rate of spatial variation of wave energy is determined as

$$\frac{1}{a} \frac{\partial a}{\partial x} = -\frac{1}{2C_g} \frac{\partial C_g}{\partial x}$$  \hspace{1cm} (42)

The group velocity for the linear Stokes wave is defined as

$$C_{gi} = \frac{\partial \omega}{\partial k_l} = C_l \left(1 + \frac{2k_l h}{\sinh 2k_l h}\right)$$  \hspace{1cm} (43)

The group velocities for the three types of Boussinesq equations are the same and given by

$$C_g = \frac{\partial \omega}{\partial k} = C \left\{1 - \frac{k^2h^2}{3(1-\alpha k^2h^2)[1-(\alpha + \frac{1}{3})k^2h^2]}\right\}$$  \hspace{1cm} (44)
For the extended Boussinesq equations, the rate of spatial variation of wave energy due to shoaling is determined by Eq. (17) not by Eq. (42). Therefore, it can be said that the group velocity given by Eq. (44) does not determine the spatial variation of wave energy over varying water depth.

3. Comparison of linear dispersion and shoaling properties in extended Boussinesq equations

Since the different value of the constant parameter $\alpha$ is employed in each set of Boussinesq equations, they produce different dispersion relation, group velocity and shoaling coefficient. Madsen and Sorensen (1992) selected $B = 1/15$ which is equivalent to $\alpha = -0.4$. Nwogu (1993) chose $\alpha = -0.39$ to minimize the errors in phase velocity in the depth range of $0 < k_0 h < \pi$, while Chen and Liu (1995) employed $\alpha = -0.3855$ for the minimal sum of errors in phase and group velocities in the same depth range as Nwogu.

Fig. 1 shows the percent errors in phase velocity produced by each set of Boussinesq equations ($= 100 \times (C - C_l) / C_l$). The analytical phase velocity $C_l$ determined by Eq. (39) is used to measure the relative errors. Fig. 2 represents the percent errors in group velocity involved in the three types of Boussinesq equations ($= 100 \times (C_g - C_{gl}) / C_{gl}$). The analytical group velocity $C_{gl}$ given by Eq. (43) for the linear Stokes wave is used to evaluate the relative errors.

From Figs. 1 and 2 it is found that the accuracy of $C$ and $C_g$ in deep water increases
Fig. 2. Percent error in group velocity versus relative water depth with fixed value of $\alpha$: ---, $\alpha = -0.4$; ----, $\alpha = -0.39$; · · · · , $\alpha = -0.3855$.

with increasing $\alpha$ value. However, the increase in $\alpha$ value leads a partial underestimation of $C$ and $C_g$ in depth range of $0.3\pi < kh < 1.0\pi$.

Fig. 3 compares the shoaling coefficient $\alpha_s$ produced by each set of extended Boussinesq equations with the analytical $\alpha_s$ of the linear Stokes wave theory (40). The $\alpha_s$ value given by Madsen and Sørensen’s equations agrees reasonably well in the range of $kh < 0.2\pi$, and starts to deviate as the water depth increases. Nwogu’s equations underestimate the $\alpha_s$ value in the range of $kh > 0.25\pi$, and this underestimation increases with increasing water depth. On the other hand, Chen and Liu’s equations overestimate the $\alpha_s$ value in the range of $kh > 1.25\pi$, and the deviation from the analytical $\alpha_s$ increases with increasing water depth. The shoaling coefficient $\alpha_s$ of Madsen and Sørensen’s equations shows the best overall accuracy in the depth range considered in this study, while that of Chen and Liu’s equations shows the largest deviation for deep water region among three types of extended Boussinesq equations.

To check a possibility of improvement of the shoaling properties in the extended Boussinesq equations, the shoaling coefficients are obtained with three different $\alpha$ values employed in each set of equations. Fig. 4(a) compares the $\alpha_s$ produced by Madsen and Sørensen’s equations with three different $\alpha$ values, i.e., $\alpha = -0.3855$, $-0.39$, and $-0.4$. As the $\alpha$ value decreases from $-0.3855$ to $-0.4$, the accuracy in $\alpha_s$ increases. Thus, the fixed value of $\alpha = -0.4$ is the best choice for the Madsen and Sørensen’s equations. As shown in Fig. 4(b) and (c) for Nwogu’s and Chen and Liu’s equations, respectively, this trend of accuracy improvement with decreasing $\alpha$ value holds for these two types of Boussinesq equations. Thus, the value of $\alpha = -0.4$ is found to be the best choice for all types of Boussinesq equations as long
4. Tuning of parameter $\alpha$ for varying water depth

In order to investigate the extendability of Boussinesq equations to deeper water for monochromatic waves, various set of tuning parameter $\alpha$ can be determined by minimizing the relative errors in a particular wave property such as phase velocity, group velocity, or shoaling coefficient for local water depth.

The optimum $\alpha$ values to get minimal squared relative errors in phase velocity, $C$, are determined analytically by equalizing the phase velocities in Eqs. (10) and (39) based on the assumption of $k = k_i$: 

$$\alpha = \frac{\tanh k_i h - k_i h + \frac{1}{3}(k_i h)^3}{(k_i h)^2(\tanh k_i h - k_i h)}$$

The $\alpha$ values are also tuned for each set of Boussinesq equations to get the most accurate group velocity, $C_g$, or shoaling coefficient, $\alpha_s$, using Eqs. (43) and (40), respectively, as target values of $\alpha$.

The phase velocity, group velocity, and shoaling coefficient calculated using each set of Boussinesq equations with the tuned values of the parameter $\alpha$ are compared
Fig. 4. Shoaling coefficient versus relative water depth with fixed value of $\alpha$: (a) Madsen and Sørensen’s equations, (b) Nwogu’s equations, (c) Chen and Liu’s equations; $-\cdot-\cdot-\cdot$, $\alpha = -0.4$; $--$, $\alpha = -0.39$; $-\cdot-\cdot-\cdot$, $\alpha = -0.3855$; $-o-o-$, linear Stokes wave theory.

with those of the linear Stokes waves. The case with $\alpha = -0.4$ is also compared together with the optimally selected cases. The value of $\alpha = -0.4$ gives the Padé $[2,2]$ approximation to the linear dispersion relation of the Stokes wave theory.

Fig. 5 shows various set of $\alpha$ values tuned for a particular wave property in each set of Boussinesq equations. For the case of $\alpha$ values tuned for phase velocity or group velocity, the $\alpha$ values for all types of the Boussinesq equations are $-0.4$ in shallow water and increase asymptotically up to $-0.334$ in very deep water with $kh > 10\pi$. For the case of shoaling coefficient, $\alpha_s$, optimum $\alpha$ values are different among the three types of the Boussinesq equations. For Madsen and Sørensen’s equations, the optimum $\alpha$ value is $-0.417$ in shallow water, increases to $-0.393$ in deeper water with $kh = 1.1\pi$, and then decreases to $-0.500$ in very deep water. For Nwogu’s equations, the optimum $\alpha$ value is $-0.414$ in shallow water, increases to $-0.406$ in deeper water with $kh = 0.73\pi$, and then decreases to $-0.419$ in very deep
Fig. 5. \( \alpha \) versus relative water depth: ---, minimal squared relative errors in \( C \); ----, minimal squared relative errors in \( C_{\alpha} \); -o-o-, minimal squared relative errors in \( \alpha_5 \) for Madsen and Sørensen’s equations; -\( \triangle \)-\( \triangle \)-, minimal squared relative errors in \( \alpha_5 \) for Nwogu’s equations; +++, minimal squared relative errors in \( \alpha_5 \) for Chen and Liu’s equations; ---, \( \alpha = -0.4 \).

water. For Chen and Liu’s equations, the optimum \( \alpha \) value is \( -0.375 \) in shallow water, increases to \( -0.334 \) in deeper water with \( k_h = 0.81\pi \), and then decreases to \( -0.465 \) in very deep water. It is interesting that the value of \( \alpha = -1/3 \) associated with the conventional Boussinesq equations of Peregrine (1967) is outside the range of the optimum values.

Fig. 6 shows the ratio \( z_{\alpha}/h \) of the elevation of horizontal velocity to the water depth associated with the chosen \( \alpha \) value. The values of \( z_{\alpha}/h \) vary with different water depth likewise the values of \( \alpha \). For the case of minimal errors in phase or group velocity, the values of \( z_{\alpha}/h \) are \( -0.553 \) in shallow water and increase asymptotically up to \( -0.424 \) in very deep water with \( k_h > 10\pi \). It is interesting that, even in very deep water, the elevation \( z_{\alpha} \) is not near the water surface but around the mid-point below the surface.

Figs. 7 and 8 show percent errors in phase velocity and group velocity, respectively, associated with various set of optimum \( \alpha \) values. The \( \alpha \) values tuned for phase velocity yields zero percent errors in the whole range of water depth. The same is true for the case of minimal errors in group velocity. These imply that the extended Boussinesq equations with a proper tuning may produce perfectly the phenomenon of variation of wave phases or energies in a constant water depth region. When the fixed value of \( \alpha = -0.4 \) is used, the errors in phase and group velocities are relatively small in comparison with the case of tuning \( \alpha \) value for proper shoaling coefficient \( \alpha_5 \).

Fig. 9 shows a comparison of the \( \alpha_5 \) values calculated using various optimum
Fig. 6. Ratio $z_a/h$ of elevation to water depth versus relative water depth: —, minimal squared relative errors in $C$; ----, minimal squared relative errors in $C_g$; -o-o-, minimal squared relative errors in $\alpha_5$ for Madsen and Sørensen’s equations; -<-$-$-, minimal squared relative errors in $\alpha_5$ for Nwogu’s equations; ++--+, minimal squared relative errors in $\alpha_5$ for Chen and Liu’s equations; -..-, $\alpha = -0.4$.

Fig. 7. Percent error in phase velocity versus relative water depth: —, minimal squared relative errors in $C$; ----, minimal squared relative errors in $C_g$; -o-o-, minimal squared relative errors in $\alpha_5$ for Madsen and Sørensen’s equations; -<-$-$-, minimal squared relative errors in $\alpha_5$ for Nwogu’s equations; ++--+, minimal squared relative errors in $\alpha_5$ for Chen and Liu’s equations; -..-, $\alpha = -0.4$. 
The \( \alpha \) values associated with each wave property and for each type of Boussinesq equation against those of the linear Stokes wave. The \( \alpha_5 \) values vary with the selection of target wave property for the tuning of \( \alpha \) value and also with the type of Boussinesq equations. For the case of \( \alpha \) values tuned for minimal errors in \( \alpha_5 \), the exact values of \( \alpha_5 \) can be obtained in the depth ranges of \( k \rho h \leq 2.6\pi \) and \( k \rho h \leq 0.81\pi \) for the equations of Madsen and Sørensen (Fig. 9(a)) and Chen and Liu (Fig. 9(c)), respectively, while the exact \( \alpha_5 \) values can be obtained in the whole range of water depth for the Nwogu’s equations (Fig. 9(b)). Considering all set of \( \alpha_5 \) values determined using various \( \alpha \) values with different targets of tuning, the range of errors in \( \alpha_5 \) given by Madsen and Sørensen’s equations are narrower than those of the other two types of Boussinesq equations. It is worthwhile to note that the fixed value of \( \alpha = -0.4 \) produces relatively smaller errors in shoaling coefficient \( \alpha_5 \) than the other \( \alpha \) values tuned for phase velocity and group velocity, regardless of the type of Boussinesq equations.

Judging from Figs. 7–9, it is found that the \( \alpha \) values tuned for a particular wave property produce relatively large errors in other wave properties when water depth becomes deep. Thus, it is impossible to determine the \( \alpha \) values yielding all the exact wave properties simultaneously.

Fig. 5 shows that the optimum values of \( \alpha \) for \( C \) and \( C_g \) are greater than −0.4, while those for \( \alpha_5 \) are less than −0.4. Thus, the fixed value of \( \alpha = -0.4 \) gives, in general, overall accuracies in phase velocity, group velocity, and shoaling coefficient for the wide range of water depth for all three types of Boussinesq equations. For the simulation of irregular waves over varying topography, it is recommended to use
the fixed value of $\alpha = -0.4$ because a wide range of frequencies is involved in the irregular waves.

5. Numerical test

Numerical experiments are conducted for horizontally one-dimensional cases using linearized Nwogu’s equations. The methods of discretization of governing equations and specification of boundary conditions are explained in detail in Lee et al. (2001). Firstly, on a constant water depth, phase velocities of monochromatic waves are measured which are generated numerically by the Boussinesq equations. The wave period is $T = 10$ sec, the still water depth is $h = 156$ m, and the relative water depth
is \( k_h = 2\pi \). The computational domain of 20 wavelengths has a wave generation point at the center of the domain and two sponge layers of thickness \( 2.5 \) times a wavelength at both ends of the domain. Three cases of \( \alpha \) values tuned for different wave properties are tested: one with minimal squared relative errors in phase velocity (case C1), one with minimal squared relative errors in group velocity (case C2), and one with \( \alpha = -0.4 \) (case C3). The phase velocity is measured as \( C = 0.99C_l (\pm 1\% \text{ error}) \), \( C = 0.81C_l (\pm 19\% \text{ error}) \), and \( C = 1.19C_l (\pm 19\% \text{ error}) \) for cases C1, C2, and C3, respectively. The measured phase velocities are almost equal to those predicted in Fig. 7.

Secondly, on a constant water depth, group velocities of bichromatic waves are measured. The mean wave period is \( T = 10\text{sec} \) (i.e., the mean wave frequency is \( f = 0.1 \text{ Hz} \)), the frequency of each component is \( f = (1 \pm 0.1)f_l \), and the still water depth is \( h = 156 \text{ m} \). Three cases of C1, C2, and C3 are tested. The group velocity is measured as \( C_g = 0.65C_l (\pm 30\% \text{ error}) \), \( C_g = 0.49C_l (\pm 2\% \text{ error}) \), and \( C_g = 0.87C_l (\pm 74\% \text{ error}) \) for cases C1, C2, and C3, respectively. The measured group velocities are almost equal to those predicted in Fig. 8.

Thirdly, on a uniformly sloping bed, amplitudes of monochromatic waves are calculated using the extended Boussinesq equations of Nwogu (1993). The wave period is \( T = 1 \text{ sec} \), the bed slope is 1:30 (V:H), and the amplitude of incident waves is \( a_o = 0.1 \text{ cm} \). The computational domain has a sloping bottom in the central part, connected by two horizontal beds with sponge layers of thickness \( 2.5 \) times a wavelength. The relative water depths of deeper and shallower horizontal beds are \( k_l h = \pi \) and \( k_l h = 0.2\pi \), respectively. The wave generation point is one wavelength apart from the starting point of the slope on the deeper horizontal bed. Three cases with different set of \( \alpha \) values are tested: one with minimal squared relative errors in shoaling coefficient (case S1), one with minimal squared relative errors in group velocity (case S2), and one with \( \alpha = -0.4 \) (case S3). Fig. 10 shows the surface elevations normalized by the amplitude of incident waves at time \( t = 30 T \). In order to check whether the variation of wave amplitudes on the sloping bed is correctly produced, the amplitudes given by the linear Stokes wave theory are plotted together. In the case of S1, the amplitude of numerically calculated waves are almost equal to that of the linear Stokes wave. In the case of S2, the error in wave amplitude becomes larger as the water becomes shallower. In the case of S3, the error becomes larger in shallower water but the magnitude of the error is not so serious in comparison with the case of S2. Fig. 10 proves that the \( \alpha \) value tuned for exact values of group velocity does not guarantee accurate production of wave shoaling.

6. Conclusion

This paper considers the choice of the tuning parameter B or \( \alpha \) in the linear version of the extended Boussinesq equations, i.e., Madsen and Sørensen (1992), Nwogu (1993), and Chen and Liu (1995). When the fixed value of \( \alpha \) is used as in the cases of Boussinesq equations selected in the present study, the accuracy of wave properties such as phase velocity, group velocity and shoaling coefficient degenerates in
general as the water depth becomes deep. For accuracy improvement in the deep water range, various sets of $\alpha$ values are determined according to a local water depth by minimizing the errors in wave properties relative to the exact ones given by the linear Stokes wave theory. As a result, a particular wave property can be produced exactly or with reasonable accuracy even for very deep water if the $\alpha$ value is tuned for it. The optimum $\alpha$ values tuned for phase velocity and group velocity increase from $-0.4$ to $-0.334$ as the water depth increases. On the contrary, those for shoaling coefficient decrease for very deep water. Thus, it may be impossible to determine the universal $\alpha$ values that produce all the wave properties accurately at the same time for the given type of Boussinesq equations whereas all the wave properties can be accurately produced by the mild-slope equations. However, the fixed value of $\alpha = -0.4$ is found to give a reasonable accuracy in all the wave properties regardless of the type of Boussinesq equations. Numerical experiments support the analytical predictions.

Fig. 10. Surface elevation of monochromatic waves on a sloping bed at time $t = 30T$: (a) $\alpha$ with minimal squared relative errors in $a_C$, (b) $\alpha$ with minimal squared relative errors in $C_g$, (c) $\alpha = -0.4$; ---, numerical solution of surface elevation; ----, exact solution of wave amplitude.
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